

Chiral Anomaly and Effective Field Theory for the Quantum Hall Liquid with Edges

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abstract

Under general assumptions, we present a low-energy effective action for the quantum Hall state when edges exist. It is shown that the chiral edge current is necessary to make the effective action to be gauge invariant. However the chiral edge current is irrelevant to the Hall current. The exactly quantized value of σ_{xy} is observed only when the Hall current does not flow at the edge region. Our effective theory is applicable to the quantum Hall liquid on a surface with non-trivial topology and physical meanings of the topology are discussed.

There are two approaches to the quantum Hall effect(QHE)[1]. In the one approach, the electric current is carried by bulk states and the boundary condition is not important. The Hall conductance is represented as a topological invariant and is quantized exactly[2]. In the other approach, the electric current is carried by edge states and the bulk effects are neglected. The Hall current is proportional to the difference of chemical potentials at the edges and the Hall conductance is proportional to a number of edge current modes[3]. Recently the relation between the bulk picture and the edge picture is discussed by many authors[4]. In this letter, we analyze a low-energy effective field theory which is applicable to the quantum Hall liquid(QHL) with edges, no matter where the Hall current flows. It will be shown that the chiral edge current is canceled in the total Hall current.

We consider that two-dimensional spinless electrons are in a perpendicular uniform magnetic field B and use the unit $\hbar = c = 1$. For a while, we suppose that electrons are bounded in negative x_2 -direction by an electrostatic potential $V(x)$. Due to $V(x)$, a diamagnetic edge current flows at the edge. We denote this current density as $J_{(0)}^\mu$ and $\partial_\mu J_{(0)}^\mu = 0$. The external field A_μ couples to electrons in a gauge invariant manner in addition to B and $V(x)$. The effective action for A_μ is obtained by integrating out the electron field. The following action should be generated by electrons in the bulk states,

$$S_{\text{CS}} = \frac{\sigma_{xy}}{2} \int d^3x f_L(x_2) \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \quad (1)$$

If f_L is constant, this is called the Chern-Simons action[5]. This action results from a parity violation in the presence of a magnetic field. f_L varies only in the edge region, $|x_2| < \delta$, $\delta = O((eB)^{-\frac{1}{2}})$, and takes constant value in other regions. We fix the value as

$$f_L(x_2) = \begin{cases} 1, & \text{for } x_2 < -\delta, \\ 0, & \text{for } x_2 > \delta. \end{cases} \quad (2)$$

When electrons in bulk states are in the quantum Hall regime, the Hall conductance is

quantized as

$$\sigma_{xy} = \frac{e^2}{2\pi} N, \quad (3)$$

where N is integer in the integer quantum Hall effect(IQHE) and is rational number in the fractional quantum Hall effect(FQHE). The electric current density is obtained from Eq.(1) as

$$J_{\text{CS}}^\mu = -\frac{\delta S_{\text{CS}}}{\delta A_\mu} = -\sigma_{xy} f_L \epsilon^{\mu\nu\rho} \partial_\nu A_\rho + \frac{\sigma_{xy}}{2} \partial_2 f_L \epsilon^{2\mu\nu} A_\nu \quad (4)$$

The second term of Eq.(4) is a edge current induced from S_{CS} .

The above argument is insufficient because the action S_{CS} is not gauge invariant in the edge region[6]. Under a gauge transformation, $A_\mu \rightarrow A_\mu + \partial_\mu \phi$, the action S_{CS} is changed by

$$\Delta S_{\text{CS}} = -\frac{\sigma_{xy}}{2} \int d^3x \partial_2 f_L \epsilon^{2\mu\nu} \partial_\mu A_\nu \phi, \quad (5)$$

where an integration of total derivative is omitted. Moreover the current density J_{CS}^μ has an anomaly,

$$\partial_\mu J_{\text{CS}}^\mu = -\sigma_{xy} \partial_2 f_L \epsilon^{2\mu\nu} \partial_\mu A_\nu. \quad (6)$$

The anomaly comes from the anomalous current in Eq.(4). Since the anomalous current is quasi-one-dimensional, the anomaly can be canceled by a quasi-one-dimensional effective action. It is noted that the quasi-one-dimensional effective action can not be obtained by A_μ -integration of anomalous current, because the anomalous current is non-integrable.

From a physical point of view, it is natural to expect that the necessary quasi-one-dimensional effective action should be generated from electrons in edge states. At the edge region, gapless electron states are exist and these states are extended in x_1 -direction. Since the edge current flows in one direction, electrons in edge states can be described by the chiral fermion and the action is

$$S_{\text{edge}}(\Psi, A) = \int d^3x (\bar{\Psi} i \gamma^i \partial_i \Psi - \frac{1}{2} g_L(x_2) \sqrt{\Delta x_2} \bar{\Psi} \gamma^i (1 - \gamma_5) \Psi A_i), \quad (7)$$

where $i = 0, 1$ and γ^i 's are Dirac matrices in $1 + 1$ dimensional spacetime. In Eq.(7), we rescale x_1 -component of vectors by a Fermi velocity v_F as $v_F \partial_1 \rightarrow \partial_1$ and $v_F A_1 \rightarrow A_1$. Hereafter we omit a Fermi velocity v_F for convenience, because our arguments are not affected by this rescaling essentially. Only the left handed fermion couples to the gauge potential A_i . In Eq.(7), Δx_2 is a short distance cutoff. This cutoff is necessary to renormalize a divergence in a current-current correlation function. Since the kinetic term does not include x_2 -derivative, one-loop diagram diverges as $\int dp_2 \sim \Lambda_2 \sim 1/\Delta x_2$. Current-current correlation function includes two vertices and its one-loop diagram behaves as (coupling) $^2 \sim \Delta x_2$. Then, we obtain a finite effective action as $\Delta x_2 \rightarrow 0$. g_L is a coordinate-dependent coupling and is determined by the gauge invariance later. The quasi-one-dimensional effective action and chiral edge current density are obtained from Eq.(7) as

$$\begin{aligned}
e^{iS_{\text{edge}}(A)} &= \int D\Psi e^{iS_{\text{edge}}(\Psi, A)}, \\
S_{\text{edge}}(A) &= \frac{1}{8\pi} \int d^3x g_L^2(x_2) A_i [\alpha g^{il} - (g^{ij} + \epsilon^{ij}) \frac{\partial_j \partial_k}{\square} (g^{kl} - \epsilon^{kl})] A_l, \\
J_L^i &= -\frac{g_L^2(x_2)}{4\pi} [\alpha g^{il} - (g^{ij} + \epsilon^{ij}) \frac{\partial_j \partial_k}{\square} (g^{kl} - \epsilon^{kl})] A_l,
\end{aligned} \tag{8}$$

where $\square = \partial^i \partial_i$ and α is regularization dependent parameter[7]. The current density J_L^i has an anomaly

$$\partial_i J_L^i = \frac{g_L^2(x_2)}{4\pi} [(1 - \alpha) \partial_i A^i - \epsilon^{ij} \partial_i A_j]. \tag{9}$$

From Eqs.(6) and (9), we obtain an anomaly free current density,

$$J_{\text{Total}}^\mu = J_{\text{CS}}^\mu + J_L^\mu + J_{(0)}^\mu, \tag{10}$$

$$\partial_\mu J_{\text{Total}}^\mu = 0,$$

where $J_{(0)}^\mu$ is a diamagnetic current due to $V(x)$, by setting

$$\alpha = 1, \quad g_L^2(x_2) = -2\pi \sigma_{xy} \partial_2 f(x_2). \tag{11}$$

In the quantum Hall regime, the coupling g_L is quantized as

$$\int_{-\infty}^{\infty} g_L^2(x_2) dx_2 = Ne^2, \quad (12)$$

using Eqs.(2) and (3). Furthermore the total effective action,

$$S_{\text{Total}} = S_{\text{CS}} + S_{\text{edge}} + \int J_{(0)}^\mu A_\mu d^3x, \quad (13)$$

becomes gauge invariant if and only if Eq.(11) is satisfied.

This anomaly cancelation was already discussed by X. G. Wen in Ref.[6]. He studied the QHL on the plane with boundary and showed the existence of the edge currents satisfying the U(1) Kac-Moody algebra. Our theory is generalization of Wen's theory to the case that the edge region has a finite width. We study not only the chiral edge current but also the current induced from the Chern-Simons action S_{CS} and show that the edge current is irrelevant to the Hall conductance.

Next we consider the case that two edges exist. Electrons are bounded in the region $-L < x_2 < 0$ by an electrostatic potential $V(x)$. In this case a function $f_L(x_2)$ is replaced with $f(x_2)$ which is given as

$$f(x_2) = \begin{cases} f_L(x_2), & \text{for } x_2 > -L/2, \\ f_R(x_2), & \text{for } x_2 < -L/2, \end{cases} \quad (14)$$

$$\text{where} \quad f_R(x_2) = \begin{cases} 1, & \text{for } x_2 > -L + \delta, \\ 0, & \text{for } x_2 < -L - \delta. \end{cases}$$

Following the previous argument of gauge invariance of S_{CS} , an effective action for the chiral fermion is required as

$$S_{\text{edge}}(\Psi, A) = \int d^3x (\bar{\Psi} i \gamma^i \partial_i \Psi - \frac{1}{2} g_L(x_2) \sqrt{\Delta x_2} \bar{\Psi} \gamma^i (1 - \gamma_5) \Psi A_i) \\ - \frac{1}{2} g_R(x_2) \sqrt{\Delta x_2} \bar{\Psi} \gamma^i (1 + \gamma_5) \Psi A_i), \quad (15)$$

$$\text{where} \quad g_R^2(x_2) = 2\pi\sigma_{xy}\partial_2 f_R(x_2).$$

Now we consider the system with two edges in the electric field $E_2(x_2)$ and use the following gauge choice,

$$E_2(x_2) = -\partial_2 A_0(x_2), \quad (16)$$

$$A_1 = A_2 = 0.$$

By substituting Eq.(16) into Eqs.(4) and (8) with Eq.(11), J_{CS}^1 and $J_L^1 + J_R^1$ are calculated as

$$\begin{aligned} J_{\text{CS}}^1 &= -\sigma_{xy}f(x_2)\partial_2 A_0 - \frac{\sigma_{xy}}{2}\partial_2 f(x_2)A_0, \\ J_L^1 + J_R^1 &= \frac{\sigma_{xy}}{2}\partial_2 f(x_2)A_0. \end{aligned} \quad (17)$$

Thus chiral edge current densities J_L^1 and J_R^1 are canceled by the edge current in J_{CS}^1 and total current density becomes

$$J_{\text{Total}}^1 = -\sigma_{xy}f(x_2)\partial_2 A_0 + J_{(0)}^\mu. \quad (18)$$

This means that the chiral edge current is irrelevant in the Hall current. From Eq.(18) the distribution of the Hall current depends on $f(x_2)$ and $E_2(x_2)$ and the total current I^1 is given by

$$\begin{aligned} I^1 &= \int_{-\infty}^{\infty} J_{\text{Total}}^1 dx_2 = \sigma_{xy}(\langle A_0 \rangle_L - \langle A_0 \rangle_R), \\ \langle A_0 \rangle_L &= - \int \partial_2 f_L(x_2) A_0 dx_2, \\ \langle A_0 \rangle_R &= + \int \partial_2 f_R(x_2) A_0 dx_2, \end{aligned} \quad (19)$$

where we assumed that $V(x)$ is so choosen that the diamagnetic current vanishes, that is $I_{(0)}^1 = \int J_{(0)}^1 dx_2 = 0$. If $f(x_2)$ is a step-like function then $|\partial_2 f(x_2)|$ is well localized normalized function in x_2 -direction. Therefore $\langle A_0 \rangle_{\text{edge}}$ is the average value of the potential at the edge region. The relation of Eq.(19) is independent of the distribution of $J_{\text{Total}}^1(x_2)$.

The current distribution is determined by the electric field E_2 in Eq.(18). Since the electric field depends on the detail of electron density and experimental settings, the

current distribution cannot be calculated by the low-energy effective action S_{Total} . What we can say from Eq.(19) is that the exactly quantized value of σ_{xy} is observed as a ratio of total current and total voltage only when the potential A_0 is constant and the Hall current does not flows at the edge region. The quantized value of σ_{xy} is observed approximately when the potential A_0 can be regarded as constant compared with a magnetic length in the edge region.

If the electrostatic potential $V(x)$ is very smooth, then $f(x)$ becomes steps-like function and many edges appear. See Fig.1. For IQHE, each step has a height $1/N$ and a number of steps is N . We denote the position of edges as R_i and L_i ($i = 1 \sim N$) at right and left sides of QHL. We assume that each edge has a width δ . Then $f(x_2)$ satisfies,

$$\begin{aligned} - \int_{L_i-\delta}^{L_i+\delta} \partial_2 f dx_2 &= \frac{1}{N}, \\ - \int_{R_i-\delta}^{R_i+\delta} \partial_2 f dx_2 &= -\frac{1}{N}. \end{aligned} \quad (20)$$

For the gauge invariance, N-chiral edge currents are needed and the relation (19) is changed as,

$$\begin{aligned} I^1 &= \sigma_{xy} \sum_{i=1}^N (\langle A_0 \rangle_{L_i} - \langle A_0 \rangle_{R_i}) / N, \\ \langle A_0 \rangle_{L_i} &= -N \int_{L_i-\delta}^{L_i+\delta} \partial_2 f(x_2) A_0 dx_2, \\ \langle A_0 \rangle_{R_i} &= +N \int_{R_i-\delta}^{R_i+\delta} \partial_2 f(x_2) A_0 dx_2. \end{aligned} \quad (21)$$

If each side of the QHL is in equilibrium, then $\langle A_0 \rangle_{L_1} = \langle A_0 \rangle_{L_2} = \dots = \langle A_0 \rangle_{L_N}$ and $\langle A_0 \rangle_{R_1} = \langle A_0 \rangle_{R_2} = \dots = \langle A_0 \rangle_{R_N}$ and Eq.(21) is equivalent to Eq.(19). But if the left side of the QHL is not in equilibrium and $\langle A_0 \rangle$'s satisfy,

$$\langle A_0 \rangle_{R_1} = \langle A_0 \rangle_{R_2} = \dots = \langle A_0 \rangle_{R_N} = \langle A_0 \rangle_R, \quad (22)$$

$$\langle A_0 \rangle_{L_1} = \langle A_0 \rangle_{L_2} = \dots = \langle A_0 \rangle_{L_K} = \langle A_0 \rangle_L,$$

$$\langle A_0 \rangle_{L_{K+1}} = \langle A_0 \rangle_{L_{K+2}} = \cdots = \langle A_0 \rangle_{L_N} = \langle A_0 \rangle_L,$$

then Eq.(21) becomes,

$$I^1 = \frac{e^2}{2\pi}(N - K)(\langle A_0 \rangle_L - \langle A_0 \rangle_R). \quad (23)$$

This anomalous IQHE was observed experimentally[8]. Same relation of Eq.(23) is also derived by using the Büttiker-Landauer formula in the edge picture. Our picture is more general than the edge picture because even if the Hall current flows only in the region between two edges in the left side region, Eq.(23) is valid.

It is interesting to consider the case that the electric field is parallel to the edges. That is

$$E_1(x_1) = -\partial_1 A_0(x_1), \quad (24)$$

$$A_1 = A_2 = 0.$$

We consider a single-step edge of Eq.(2) here. The edge current density in J_{CS}^μ is the second term in Eq.(4), and the edge current $I_{\text{CS,edge}}^i$ and I_L^i are given by

$$I_{\text{CS,edge}}^i = \int J_{\text{CS,edge}}^i dx_2 = \frac{\sigma_{xy}}{2} \int \partial_2 f \epsilon^{ij} A_j dx_2 = -\frac{\sigma_{xy}}{2} \epsilon^{ij} A_j, \quad (25)$$

$$I_L^i = \int J_L^i dx_2, \quad (i = 0, 1).$$

Using Eqs.(24), (25), and (9) with Eq.(11), we obtain

$$\partial_i I_{\text{CS,edge}}^i = -\frac{\sigma_{xy}}{2} E_1, \quad (26)$$

$$\partial_i I_L^i = -\frac{\sigma_{xy}}{2} E_1.$$

Thus the total edge current, $I_{\text{Total,edge}}^i = I_{\text{CS,edge}}^i + I_L^i$, satisfies

$$\partial_i I_{\text{Total,edge}}^i = -\sigma_{xy} E_1. \quad (27)$$

We can easily show that $I_{\text{Total,edge}}^1$ vanishes. Then Eq.(27) means that the current flowing perpendicular to the edge equals to $-\sigma_{xy}E_1$ and the QHE occurs.

In the case of Eq.(16), it is important that anomalies of J_{CS} and J_L cancel. On the other hand, in the case of Eq.(24), sum of each anomaly contributes to the Hall current in Eq.(27).

Using Eq.(27), we can calculate the charge of a quasiparticle moving on edges in the excited states[6]. Excited states can be constructed by adding a fictitious flux $\phi_0 = 2\pi/e$ to the bulk states. Using the Bianchi identity, $\epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho} = 0$, a charge of quasiparticle is given by

$$Q = \int dt dx_1 \partial_0 I_{\text{Total,edge}}^0 = - \int dt dx_1 \sigma_{xy} E_1 = \sigma_{xy} \int dt \partial_0 \phi = \sigma_{xy} \phi_0 = Ne. \quad (28)$$

In FQHE, N takes fractional value and the quasiparticle has a fractional charge. In order for the system to be not changed by insertion of flux ϕ_0 at x_0 , $f(x)$ in S_{CS} has to vanish at x_0 . This corresponds to the quasiparticle in the bulk states[9] which has a charge $-Ne$. These excitations would have complex dynamics and can not be described by our low-energy effective field theory. To describe these excitations, we have to treat $f(x)$ in S_{CS} as a dynamical quantity. Then we should regard $f(x)$ as an order parameter for topological states of QHL.

We can imagine a two-dimensional phase diagram for f and ∂f . A line of $f = 0$ is a parity unbroken phase and a line of $\partial f = 0$ is a topological phase. There are stable points on the topological phase line. These points correspond to QHL states. The QHL with edges is represented as a path linking these stable points. The path is parametrized by x_2 , $-\infty < x_2 < \infty$. Fig.2 shows a single-step case(P_1) and a double-step case(P_2) for $N=2$ IQHE.

Finally we generalize previous arguments to the QHL on a surface with non-trivial

topology. That is a surface with boundaries, punctures, and handles. For simplicity we take $\delta = 0$ and edges of QHL can be regarded as boundaries of two-dimensional surface. Since the action S_{CS} is diffeomorphism invariant without metric of the surface (topological invariant), we can deform the surface as boundaries to become straight lines. Thus previous arguments on the edge are valid in the case of curved edges. A handle is regarded as a cylinder attached to two boundaries of the surface and previous arguments of two edges are also applicable to it. Each handle has a pair of non-contractable loops l_1, l_2 . Without changing a magnetic field on the surface, fluxes ϕ_1, ϕ_2 can go through the loops. The flux and the current flowing across the loop l_i are related as

$$I_i = \sigma_{xy} \int_{l_i} \epsilon^{ij} E_j = \sigma_{xy} \dot{\phi}_i, \quad (i = 1, 2). \quad (29)$$

Thus a handle works as a current source and a current drain. A puncture is an infinitesimal hole of the surface and is represented as a point where $f(x)$ vanishes in our theory. Without changing a magnetic field on the surface, unit flux can be inserted into the puncture. Then a puncture is geometric representation of a quasiparticle in the bulk states.

In conclusion, we obtained a low-energy effective theory for the QHL with edges. Although chiral edge currents are necessary for the gauge invariance of the effective action, it does not contribute to the Hall current. It is shown that the quantization of σ_{xy} is observed exactly only when the Hall current does not flow in the edge regions. The anomalous IQHE was explained in our theory. Using a function f as an order parameter, we presented a phase diagram for the QHL with edges. Physical meanings of topology of the surface where electrons live were clarified.

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Figure Caption

Fig.1 : $f(x)$ with two steps for the N=2 IQHE.

Fig.2 : A phase diagram for the N=2 QHL with one step(P_1) and two steps(P_2). The paths in the upper half plane represent right side edges of the QHL and paths in the lower half plane represent left side edges of the QHL.